Hw 8.

Let $f : [a,b] \to \mathbb{R}$ and $\pi := \{a = x_0 < x_1 < \cdots < x_n = b\} \in par[a,b]$, a partition of $[a,b]$. Define

$$t(f;\pi) = \sum_{i=1}^{n} |f(x_i) - f(x_{i-1})|$$

and

$$T_a^b(f) := \sup\{t(f;\pi) : \pi \in par[a,b]\} \quad (\le +\infty)$$

Q1. Show that $t(f;\pi) \uparrow_\pi$: if partitions $\pi \subseteq \pi'$ then

$$0 \le t(f;\pi) \le t(f;\pi')$$

Q2 Show that $T_a^b(f) = T_a^c(f) + T_c^b(f) \ \forall c \in (a,b)$.
(so $T_a^c(f) \uparrow_c$).

Q3. Do Q1, Q2 similarly for $p(t,\pi) = \sum_{i=1}^{n} [f(x_i) - f(x_{i-1})]^+$ and $P_a^b(f) := \sup\{p(t;\pi) : \pi \in par[a,b]\}$.

Also for $n(t,\pi)$ and $N_a^b(f)$ (with $[f(x_i) - f(x_{i-1})]^-$ in place of $[f(x_i) - f(x_{i-1})]^+$).

Q4 Show $T_a^b(f) = P_a^b(f) + N_a^b(f)$. Hint: Let $\pi_1, \pi_2 \in p[a,b]$. Let $\pi_1 \cup \pi_2 \in p[a,b]$ consisting of all partition points of π_1 and π_2. Then, since $|a| = a^+ + a^-$ $\forall a \in \mathbb{R}$,

$$T_a^b(f) \ge t(f;\pi_1 \cup \pi_2) = p(f,\pi_1 \cup \pi_2) + n(f,\pi_1 \cup \pi_2) \ge p(f,\pi_1) + n(f,\pi_2);$$

since this is true $\forall \pi_1, \pi_2 \in par[a,b]$, it follows that

$$T_a^b(f) \ge P_a^b(f) + N_a^b(f).$$

The opposite ineq is easy.

Q5. Show that $f(b) - f(a) = P_a^b(f) - N_a^b(f)$ if $N_a^b(f) \in \mathbb{R}$

Q6. Let $BV[a,b] \ni f$: $T_a^b(f) < +\infty$. Then $P_a^x(f), N_a^x(f)$ $(x \in [a,b])$ are $\uparrow$-functions from $[a,b]$ into $\mathbb{R}$ and $f(x) = f(a) + P_a^x(f) - N_a^x(f)$ $\forall x \in [a,b]$.

Q7. Let $f \in ABC[a,b]$. Show that $f \in BV[a,b]$: Let $\varepsilon := 1$, and take $\delta > 0$ accordingly to the definition of absolute continuity of f. Take $N \in \mathbb{N}$ such that $\frac{b-a}{N} < \delta$ and divide $[a,b]$ into N-many subintervals of equal length with partition points

$$a = x_0 < x_1 < x_2 < \cdots < x_{N-1} < x_N = b.$$

Show that $T_{x_{i-1}}^{x_i}(f) \leq \varepsilon = 1$ and so $T_a^b(f) \leq N$.

Q8. Let $0 \leq f \in \mathcal{L}[a,b]$ and let $F(x) = \int_a^x f \;\forall x \in [a,b]$. Show that $F \in ABC[a,b]$. Can you drop the condition $f \geq 0$? (Yes $\leftarrow f = f^+ - f^-$; also $F_1, F_2 \in ABC[a,b] \Rightarrow F_1 \pm F_2 \in ABC[a,b]$).

Q9. Let $f : [a,b] \to \mathbb{R}$ be $\uparrow$ (so $\downarrow$). Show that $f^{-1}(I)$ whenever I is an interval and hence f is measurable. (Hint: use the characteristic property for an interval: — order-convexity).